CONNECTIVITY AND COHOMOLOGY

<u>Slogan</u>: "High connectivity of simplicial complexes
$\implies$ Vanishing Group Cohomology"

<u>Goal</u>: Farey graph path connected

$$\implies H^1(SL_2\mathbb{Z};\mathbb{Q}) = 0$$

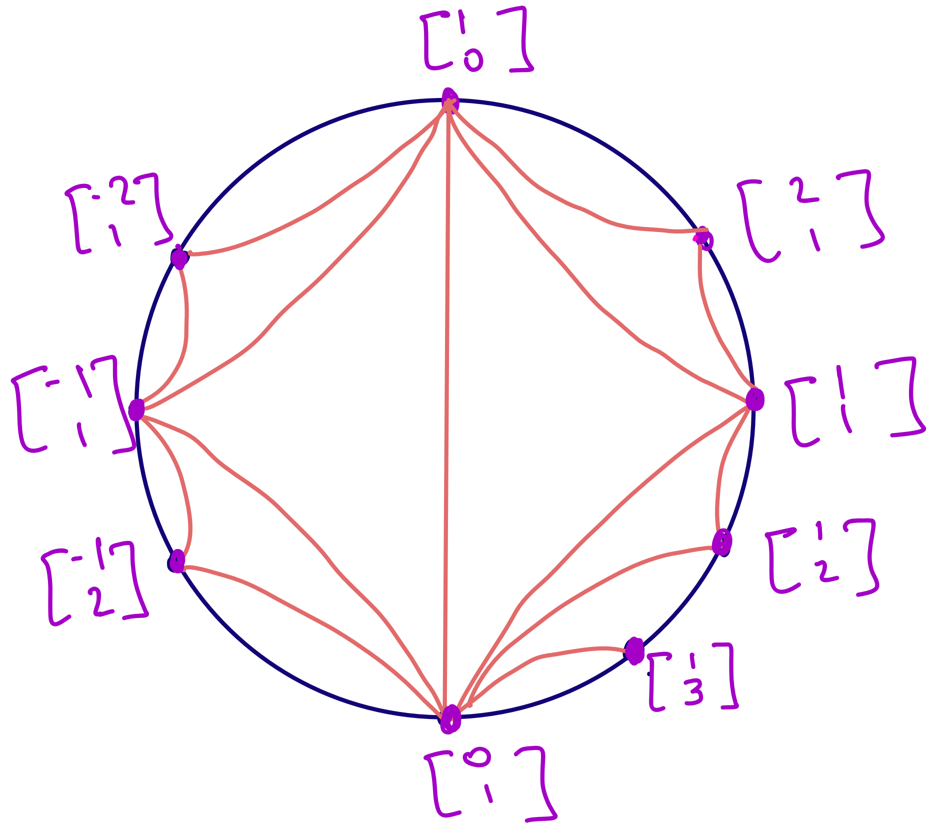
- <u>Farey Graph</u>:

Vertices $\longleftrightarrow$ $\mathbb{Q} \cup \{\infty\}$

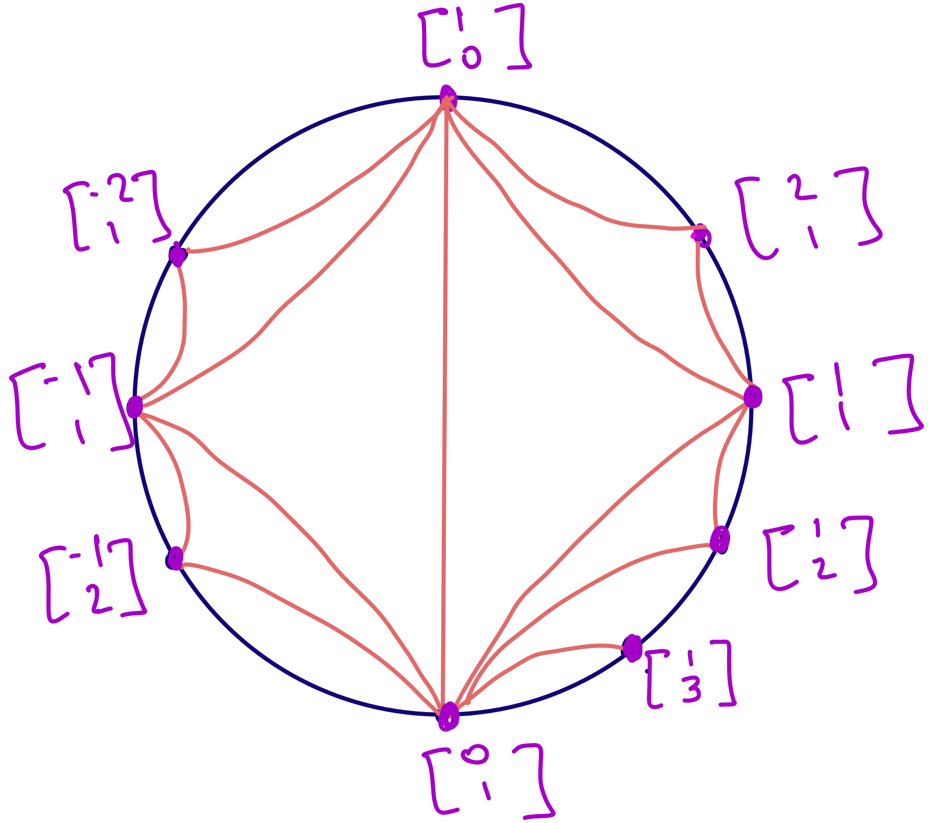
or $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{Z}^2$

in lowest form

Edges $\longleftrightarrow$ $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}$ $\mathbb{Z}^2$-basis



- **Farey Graph:**



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- $SL_2\mathbb{Z} \curvearrowright \mathbb{Q} \cup \{\infty\}$

$M \curvearrowright \begin{bmatrix} a \\ b \end{bmatrix} = M \begin{bmatrix} a \\ b \end{bmatrix}$

$SL_2\mathbb{Z} \curvearrowright \left\langle \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle$ $\left(= H_0(\mathbb{Q} \cup \{\infty\}) \right)$

<u>Thm</u> (Borel-Serre, Bieri-Eckmann)

$$H^1(SL_2\mathbb{Z};\mathbb{Q}) = H_0(\mathbb{Q}\cup\{\infty\})_{SL_2\mathbb{Z}}$$

$$\left(:= \frac{H_0(\mathbb{Q}\cup\{\infty\})}{\langle A - gA \mid g\in SL_2\mathbb{Z}\rangle} \right)$$

<u>Eg</u>: $A = \begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}$

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\left(\begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix}$$

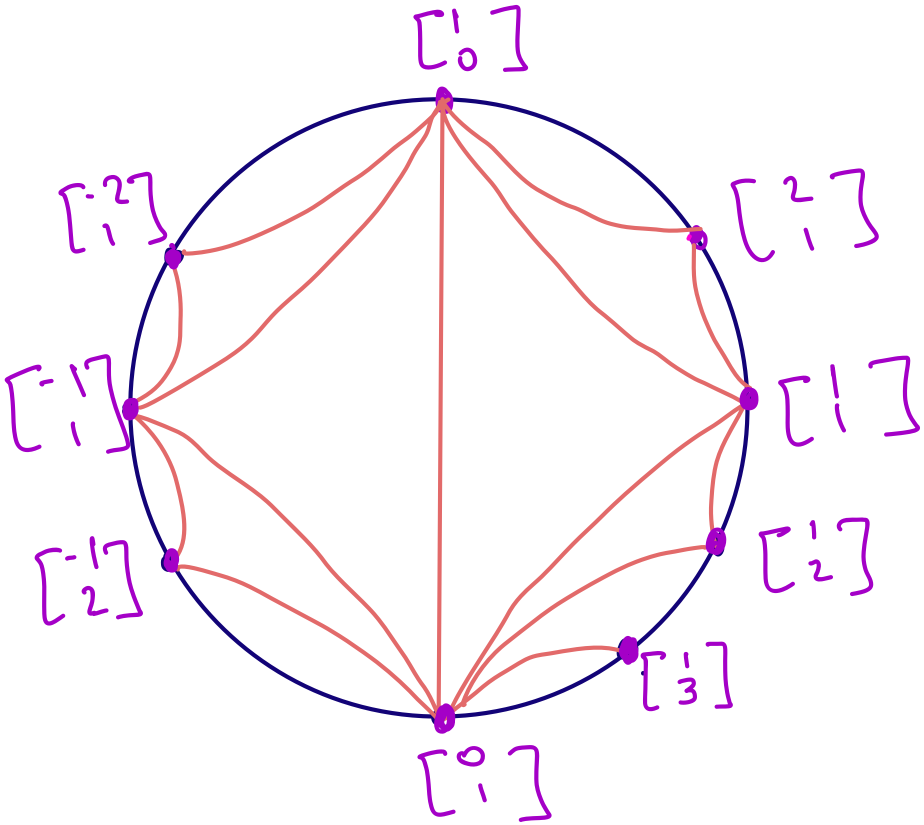
In coinvariants, $A = -A$

so $A = 0$

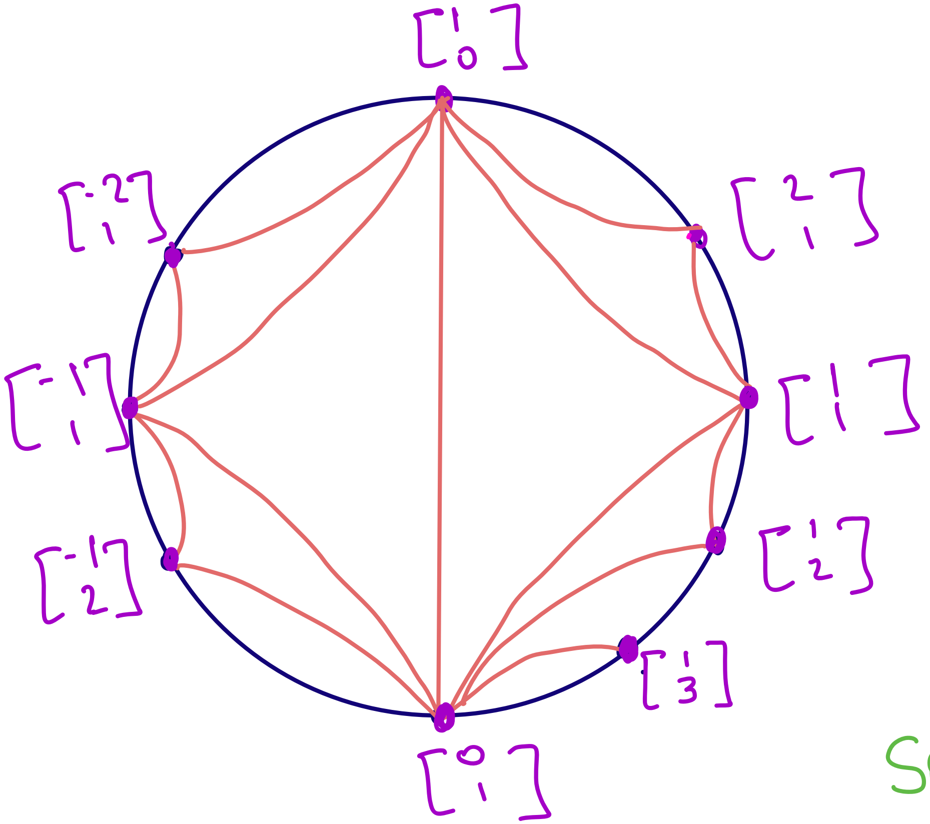
→ Can do this if $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}$ is a $\mathbb{Z}^2$-basis

(i.e., an edge in Farey!)

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1}\left(\begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix}\right) = \begin{bmatrix} c \\ d \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix}$$



Q: What about non-edges?



Eg: $\begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}1\\2\end{bmatrix}$

$$= \overbrace{\left(\begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix}\right) + \left(\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\2\end{bmatrix}\right)}^{\text{path in Farey}}$$

$= 0$ in coinvariants

So $H^1(SL_2\mathbb{Z};\mathbb{Q}) = 0$